Structure-Preserving Finite Element Methods for Fluids

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Aim

What is the role of conserved quantities in fluid dynamics simulations?

Conservation in Navier–Stokes

Let $\Omega \subset \mathbb{R}^n$ be a domain. Incompressible Navier–Stokes:

$$\partial_t u + (u \cdot \nabla)u = \mathrm{Re}^{-1}\Delta u - \nabla p + f,$$
$$\nabla \cdot u = 0$$

with periodic or zero boundary conditions. Assuming no external force and taking "infinite Reynolds number" (zero viscosity), we get the Euler momentum equation

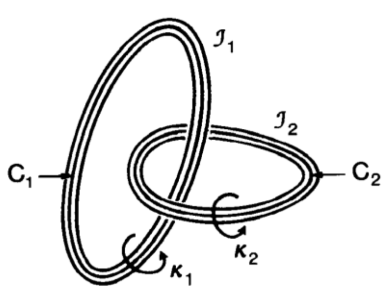
$$\partial_t u + (u \cdot \nabla)u = -\nabla p.$$

Define energy and helicity:

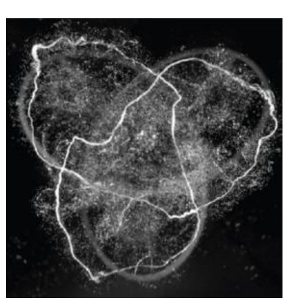
$$E := \frac{1}{2}\int_\Omega u \cdot u \, \mathrm{d}x,$$
$$\mathcal{H} := \int_\Omega u \cdot (\nabla \times u) \, \mathrm{d}x.$$

If u is a solution to incompressible Euler equations, these are **conserved in time**, i.e. $\mathrm{d}E/\mathrm{d}t = 0 = \mathrm{d}\mathcal{H}/\mathrm{d}t$.

What is helicity? One interpretation: quantifies "knottedness" of vortex lines: below left, $\mathcal{H} = \pm 2n\kappa_1\kappa_2$ where n is the linking number of the vortex loops.
Aside: knotted vortex rings have been created experimentally!



Source: (1), Figure 1



Source: (2), Figure 1

The Finite Element Method

Take Poisson problem as an example: given f, find smooth u such that

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Our goal: find some u_h such that $u_h \to u$ as $h \to 0$.

First find variational formulation: if u solves Poisson problem then for all smooth v with value zero on $\partial\Omega$

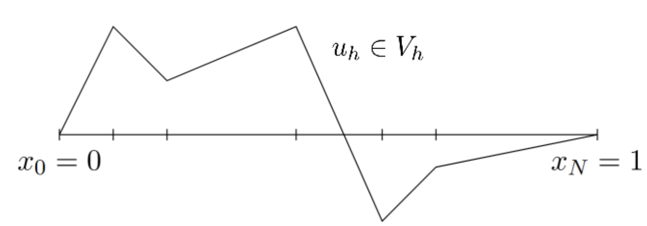
$$-\int_\Omega (\Delta u) v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x.$$

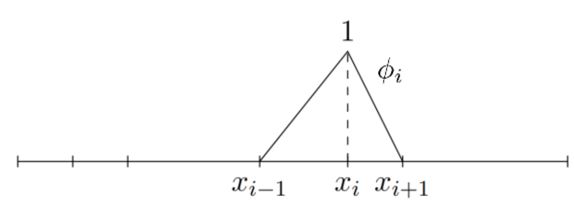
Integrate by parts to turn LHS into $\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$. Now both sides are linear in u and v, so we can abstract to: find $u \in V$ such that

$$a(u, v) = (f, v) \quad \forall v \in V.$$

For our case, $V = H_0^1(\Omega)$. An issue: this is infinite-dimensional, so it is difficult to solve variational problem. Solution (**Galerkin method**): replace V with a finite-dimensional $V_h \subset V$.

The **finite element method**: partition Ω into simplices (intervals, triangles, tetrahedra, ...). Take V_h to consist of functions that are polynomials of some fixed degree, with "hat" basis functions:





Since $u_h = \sum_{i=1}^d c_i\phi_i$, Galerkin problem becomes: find $\mathbf{c} = (c_1, \ldots, c_d)$ such that

$$\sum_{i=1}^d a(\phi_i, \phi_j)c_i = (f, \phi_j) \quad \text{for all } j = 1, \ldots, d,$$

i.e. solve $A\mathbf{c} = \mathbf{F}$.

Preserving structures in FEM

Key point: for finite element approximation of Euler equations, E_h and $\mathcal{H}_h$ are not necessarily conserved. However, Rebholz 2007 (3) designed numerical schemes that conserved both (replace nonlinear term with $\omega \times u$, modify pressure, trapezoidal timestep, solve for 3 fields).

Some further areas of research:

- Computational tradeoff? I.e. when is helicity conservation "worth it"?
- How does helicity affect dynamics (helicity cascade)?
- General manifolds (FEEC)
- Domains with nontrivial (co)homology
- Turbulence models
- Magnetohydrodynamics

References & acknowledgements

(1) Moffat & Tsinober, Helicity in Laminar and Turbulent Flow, *Ann. Rev. Fluid Mech.* 1992

(2) Kleckner & Irvine, Creation and Dynamics of Knotted Vortices, *Nature Physics,* 2013

(3) Rebholz, An Energy- and Helicity-Conserving Finite Element Scheme for the Navier–Stokes Equations, *SIAM J. Numer. Anal.* 2007

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